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Asymptotics of eigenvalues of the Aharonov–Bohm operator with a strong δ-interaction on a loop

G Honnouvo[1,2,3] **and M N Hounkonnou**[1,2]

[1] International Chair in Mathematical Physics and Applications (ICMPA), 072 BP 50, Cotonou, Benin
[2] Unité de Recherche en Physique Théorique (URPT), Institut de Mathématiques et de Sciences Physiques (IMSP), 01 BP 2628 Porto-Novo, Benin
[3] Department of Mathematics and Statistics, Concordia University, 7141 Sherbrooke St West, Montreal, Quebec H4B 1R6, Canada

E-mail: g_honnouvo@yahoo.fr and hounkonnou1@yahoo.fr

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Abstract
We investigate the two-dimensional Aharonov–Bohm operator $H_{c_0,\beta} = (-\mathrm{i}\nabla - A)^2 - \beta\delta(\cdot - \Gamma)$, where Γ is a smooth loop and A is the vector potential which corresponds to the Aharonov–Bohm potential. The asymptotics of negative eigenvalues of $H_{c_0,\beta}$ for $\beta \longrightarrow +\infty$ is found. We also prove that for large enough positive value of β the system exhibits persistent currents.

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1. Introduction

In the presence of a static magnetic field, a single isolated normal-metal loop is predicted to carry an equilibrium current [1], which is periodic in the magnetic flux Φ threading the loop. This current arises due to the boundary conditions [2] imposed by the doubly connected nature of the loop. As a consequence of these boundary conditions, the free energy E and the thermodynamic current $I(\Phi) = \frac{\partial E}{\partial \Phi}$ are periodic in Φ, with a fundamental period $\Phi_0 = \hbar/e$. In recent papers [3, 4] Exner and Yoshitomi have derived an asymptotic formula showing that if the δ-coupling is strong or in a homogeneous magnetic field B perpendicular to the plane, the negative eigenvalues approach those of the ideal model in which the geometry of Γ is taken into account by means of an effective curvature-induced potential. The purpose of this paper is to ask a similar question in a situation when the electron is subject to a Bohm–Aharonov potential. We are going to derive an analogous asymptotic formula where the presence of the magnetic field is taken into account via the boundary conditions specifying the domain of the comparison operator as in [4]. As a consequence of this result, we prove that the system exhibits persistent currents.

2. The model and the results

In this section, we study the Aharonov–Bohm operator in $L^2(\mathbb{R}^2)$ with an attractive δ-interaction applied to a loop. We use the gauge field $A = c_0\left(\frac{-y}{x^2+y^2}; \frac{x}{x^2+y^2}\right)$. Let $\Gamma : [0, L] \ni s \mapsto (\Gamma_1(s), \Gamma_2(s)) \in \mathbb{R}^2$ be the closed counter-clockwise C^4 Jordan curve which is parametrized by its arc length. Given $\beta > 0$ and $c_0 \in]0, 1[$, we define the quadratic form

$$q_{c_0,\beta}(f; f) = \left\| \left(-\mathrm{i}\partial_x + \frac{c_0 y}{x^2+y^2}\right) f \right\|^2_{L^2(\mathbb{R}^2)} + \left\| \left(-\mathrm{i}\partial_y - \frac{c_0 x}{x^2+y^2}\right) f \right\|^2_{L^2(\mathbb{R}^2)} - \beta \int_\Gamma |f(x)|^2 \,\mathrm{d}s$$

with the domain $H^1(\mathbb{R}^2)$, where $\partial_x \equiv \frac{\partial}{\partial_x}$, and the norm refers to $L^2(\mathbb{R}^2)$.

Let us denote by $H_{c_0,\beta}$ the self-adjoint operator associated with the form $q_{c_0,\beta}(\,,\,)$:

$$H_{c_0,\beta} = (-\mathrm{i}\nabla - A)^2 - \beta\delta(. - \Gamma).$$

Our main goal is to study, as in [4], the asymptotic behaviour of the negative eigenvalues of $H_{c_0,\beta}$ as $\beta \longrightarrow +\infty$.

Let $\gamma : \mathbb{R} \longrightarrow \mathbb{R}$ be the signed curvature of Γ , i.e.

$$\gamma(s) := (\Gamma_1''\Gamma_2' - \Gamma_2''\Gamma_1')(s).$$

Next we need a comparison operator on the curve

$$S_{c_0} = -\frac{\mathrm{d}^2}{\mathrm{d}s^2} - \frac{1}{4}\gamma(s)^2 \text{ in } L^2((0; L)) \tag{2.1}$$

with the domain

$$P_{c_0} = \{u \in H^2(]0; L[); u^{(k)}(L) = u^{(k)}(0); k = 1, 2\}. \tag{2.2}$$

For $j \in \mathbb{N}$, we denote by $\mu_j(c_0)$ the jth eigenvalue of the operator S_{c_0} counted with multiplicity. This allows us to formulate our main result and the proof follows using the same method as in [4]:

Theorem 2.1. *Let n be an arbitrary integer and I be a non-empty compact subset of* $]0, 1[$. *Then there exists $\beta(n, I)$ such that $\#\{\sigma_d(H_{c_0,\beta}) \cap]-\infty, 0[\} \geqslant n$ for $\beta \geqslant \beta(n, I)$ and $c_0 \in I$.*

For $\beta \geqslant \beta(n, I)$ and $c_0 \in I$ we denote by $\lambda_n(c_0, \beta)$ the nth eigenvalue of $H_{c_0,\beta}$ counted with multiplicity.

Then $\lambda_n(c_0, \beta)$ admits an asymptotic expansion of the form $\lambda_n(c_0, \beta) = -\frac{1}{4}\beta^2 + \mu_n(c_0) + \mathcal{O}(\beta^{-1} \ln \beta)$ as $\beta \to +\infty$; where the error term is uniform with respect to $c_0 \in I$.

The existence of persistent currents is given as a consequence of the following result.

Corollary 2.1. *Let $n \in \mathbb{N}$. Then there exists a constant $\beta_1(n, I) > 0$ such that the function $\lambda_n(\cdot, \beta)$ is not constant for $\beta > \beta_1(n, I)$.*

Since the spectral properties of $H_{c_0,\beta}$ are clearly invariant with respect to Euclidean transformation of the plane, we may assume without any loss of generality that the curve Γ parametrizes in the following way:

$$\Gamma_1(s) = \Gamma_1(0) + \int_0^s \cos H(t)\,\mathrm{d}t \qquad \Gamma_2(s) = \Gamma_2(0) + \int_0^s \sin H(t)\,\mathrm{d}t$$

where $H(t) \equiv -\int_0^t \gamma(u)\,\mathrm{d}u$. Let Ψ_a be the map

$$\Psi_a : [0, L) \times (-a, a) \ni (s, u) \mapsto (\Gamma_1(s) - u\Gamma_2'(s), \Gamma_2(s) + u\Gamma_1'(s)) \in \mathbb{R}^2.$$

From [3] we know that there exists $a_1 > 0$ such that the map Ψ_a is injective for all $a \in (0, a_1]$. We thus fix $a \in (0, a_1)$ and denote by Σ_a the strip of width $2a$ enclosing Γ

$$\Sigma_a \equiv \Psi_a([0, L) \times (-a, a)).$$

Then the set $\mathbb{R}^2/\Sigma_a$ consists of two connected components which we denote by $\wedge_a^{\text{in}}$ and $\wedge_a^{\text{out}}$, where the interior one, $\wedge_a^{\text{in}}$, is compact. We define a pair of quadratic forms,

$$q^{\pm}_{c_0,a,\beta}(f; f) = \left\| \left(-\mathrm{i}\partial_x + \frac{c_0 y}{x^2 + y^2}\right) f \right\|^2_{L^2(\Sigma_a)} + \left\| \left(-\mathrm{i}\partial_y - \frac{c_0 x}{x^2 + y^2}\right) f \right\|^2_{L^2(\Sigma_a)} - \beta \int_\Gamma |f(x)|^2 \, \mathrm{d}s$$

which are given by the same expression but differ by their domains, the latter in $H_0^1(\Sigma_a)$ for $q^{+}_{c_0,a,\beta}$ and $H^1(\Sigma_a)$ for $q^{-}_{c_0,a,\beta}$. Furthermore, we introduce the quadratic forms

$$e^{\pm}_{c_0,a}(f; f) = \left\| \left(-\mathrm{i}\partial_x + \frac{c_0 y}{x^2 + y^2}\right) f \right\|^2_{L^2(\wedge_a^j)} + \left\| \left(-\mathrm{i}\partial_y - \frac{c_0 x}{x^2 + y^2}\right) f \right\|^2_{L^2(\wedge_a^j)} \tag{2.3}$$

for $j =$ out, in, with the domains $H_0^1\left(\wedge_a^j\right)$ and $H^1\left(\wedge_a^j\right)$ corresponding to the $\pm$ sign, respectively. Let $L^{\pm}_{c_0,a,\beta}$, $E^{\text{out},\pm}_{c_0,a}$ and $E^{\text{in},\pm}_{c_0,a}$ be the self-adjoint operators associated with the forms $q^{\pm}_{c_0,a,\beta}$, $e^{\text{out},\pm}_{c_0,a}$ and $e^{\text{in},\pm}_{c_0,a}$, respectively.

As in [3] we are going to use the Dirichlet–Neumann bracketing with additional boundary conditions at the boundary of Σ_a. One can easily see this by comparing the form domains of the involved operators, cf [4] or [5, theorem XIII.2]. We obtain

$$E^{\text{in},-}_{c_0,a} \oplus L^{-}_{c_0,a,\beta} \oplus E^{\text{out},-}_{c_0,a} \leqslant H_{c_0,a} \leqslant E^{\text{in},+}_{c_0,a} \oplus L^{+}_{c_0,a,\beta} \oplus E^{\text{out},+}_{c_0,a} \tag{2.4}$$

with the decomposed estimating operators in $L^2(\mathbb{R}^2) = L^2\left(\wedge_a^{\text{in}}\right) \oplus L^2(\Sigma_a) \oplus L^2\left(\wedge_a^{\text{out}}\right)$. In order to assess the negative eigenvalues of $H_{c_0,\beta}$, it suffices to consider those of $L^{+}_{c_0,a,\beta}$ and $L^{-}_{c_0,a,\beta}$, because the other operators involved in (2.4) are positive. Since the loop is smooth, we can pass inside Σ_a to the natural curvilinear coordinates. We state

$$(U_a f)(s, u) = (1 + u\gamma(s))^{1/2} f(\Psi_a(s, u)) \qquad \text{for } f \in L^2(\Sigma_a)$$

which defines the unitary operator U_a from $L^2(\Sigma_a)$ to $L^2((0, L) \times (-a, a))$. To express the estimating operators in the new variables, we introduce

$$\mathcal{Q}_a^+ = \{\psi \in H^1((0, L) \times (-a, a));\ \psi(L, \cdot) = \psi(0, \cdot) \text{ on } (-a, a);\ \psi(\cdot, a) = \psi(\cdot, -a) \text{ on } (0, L)\}$$
$$\mathcal{Q}_a^- = \{\psi \in H^1((0, L) \times (-a, a));\ \psi(L, \cdot) = \psi(0, \cdot) \text{ on } (-a, a)\}$$

and define the quadratic forms

$$\begin{aligned} b^{\pm}_{c_0,a,\beta}[g] = {} & \int_0^L \int_{-a}^a (1 + u\gamma(s))^{-2} |\partial_s g|^2 \, \mathrm{d}u \, \mathrm{d}s + \int_0^L \int_{-a}^a |\partial_u g|^2 \\ & + \int_0^L \int_{-a}^a V(s, u) |g|^2 \, \mathrm{d}s \, \mathrm{d}u - \beta \int_0^L |g(s, 0)|^2 \, \mathrm{d}s \\ & - \frac{b_\pm}{2} \int_0^L \frac{\gamma(s)}{1 + a\gamma(s)} |g(s, a)|^2 \, \mathrm{d}s + \frac{b_\pm}{2} \int_0^L \frac{\gamma(s)}{1 - a\gamma(s)} |g(s, -a)|^2 \, \mathrm{d}s \\ & + c_0^2 \int_0^L \int_{-a}^a \theta(s, u) |g|^2 \, \mathrm{d}u \, \mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&+ 2c_0\, \mathrm{Im} \int_0^L \int_{-a}^{a} \theta(s,u)(\Gamma_2 + u\Gamma_1')((1+u\gamma)^{-1} \cos H \overline{g} \partial_s g - \sin H \overline{g} \partial_u g)\, \mathrm{d}u\, \mathrm{d}s \\
&- 2c_0\, \mathrm{Im} \int_0^L \int_{-a}^{a} \theta(s,u)(\Gamma_1 - u\Gamma_2')((1+u\gamma)^{-1} \sin H \overline{g} \partial_s g + \cos H \overline{g} \partial_u g)\, \mathrm{d}u\, \mathrm{d}s
\end{aligned} \tag{2.5}
$$

on $\mathcal{Q}_a^{\pm}$ respectively, where

$V(s,u) = \frac{1}{2}(1+u\gamma(s))^{-3}u\gamma(s)'' - \frac{5}{4}(1+u\gamma(s))^{-4}u^2\gamma'(s)^2 - \frac{1}{4}(1+u\gamma(s))^{-2}\gamma(s)^2$

$\theta(s,u) = (\Gamma_1^2(s) + \Gamma_2^2(s) + u^2 - 2u(\Gamma_1(s)\Gamma_2'(s) - \Gamma_2(s)\Gamma_1'(s)))^{-1}$

$b_+ = 0$ and $b_- = 1$.

Let $D^{\pm}_{c_0,a,\beta}$ be the self-adjoint operators associated with the forms $b^{\pm}_{c_0,a,\beta}$, respectively. By analogy with [3], we get the following result.

Lemma 2.1. $U_a D^{\pm}_{c_0,a,\beta} U_a = L^{\pm}_{c_0,a,\beta}$.

In order to eliminate the coefficients of $\overline{g}\partial_s g$ and $\overline{g}\partial_u g$ in (2.5) modulo small errors, we employ the following unitary operator:

$$(M_{c_0}h)(s,u) = \exp[\mathrm{i}K(s,u)]h(s,u). \tag{2.6}$$

Replacing $M_{c_0}h$ in (2.5), it becomes

$$
\begin{aligned}
c^{\pm}_{c_0,a,\beta}[g] = &\int_0^L \int_{-a}^{a} (1+u\gamma(s))^{-2}|g_s|^2\, \mathrm{d}u\, \mathrm{d}s + \int_0^L \int_{-a}^{a} |g_u|^2\, \mathrm{d}u\, \mathrm{d}s \\
&- \beta \int_0^L |g(s,0)|^2\, \mathrm{d}s - \frac{b_\pm}{2} \int_0^L \frac{\gamma(s)}{1+a\gamma(s)} |g(s,a)|^2\, \mathrm{d}s \\
&+ \frac{b_\pm}{2} \int_0^L \frac{\gamma(s)}{1-a\gamma(s)} |g(s,-a)|^2\, \mathrm{d}s \\
&+ \int_0^L \int_{-a}^{a} \big(\theta(s,u)c_0^2 + (1+u\gamma(s))^{-2}K_s^2 + K_u^2 + V(s,u) \\
&+ 2c_0\Omega_1(s,u)K_s - 2c_0\Omega_2(s,u)K_s\big)|g|^2\, \mathrm{d}u\, \mathrm{d}s \\
&+ 2\, \mathrm{Im} \int_0^L \int_{-a}^{a} (c_0\Omega_1(s,u) + (1+u\gamma(s))^{-2}K_s)\overline{g} g_s\, \mathrm{d}u\, \mathrm{d}s \\
&- 2\, \mathrm{Im} \int_0^L \int_{-a}^{a} (c_0\Omega_2(s,u) - K_u)\overline{g} g_u\, \mathrm{d}u\, \mathrm{d}s
\end{aligned} \tag{2.7}
$$

where

$$\Omega_1(s,u) = \theta(s,u)(\Gamma_2 \cos H - \Gamma_1 \sin H + u)(1+u\gamma)^{-1} \tag{2.8}$$

$$\Omega_2(s,u) = \theta(s,u)(\Gamma_1 \cos H + \Gamma_2 \sin H)(1+u\gamma)^{-1} \tag{2.9}$$

$$K_s = \frac{\partial_s K(s,u)}{\partial s} \qquad K_u = \frac{\partial_u K(s,u)}{\partial u} \qquad g_s = \frac{\partial_s g(s,u)}{\partial s} \qquad g_u = \frac{\partial_u g(s,u)}{\partial u}. \tag{2.10}$$

To eliminate the coefficients of $\overline{g}\partial_u g$ in $c^{\pm}_{c_0,a,\beta}[g]$, we have the following differential equation:

$$\frac{\partial K(s,u)}{\partial_u} = c_0\Omega_2(s,u) \tag{2.11}$$

and then, we have

$$K(s,u) = \int_0^u c_0\Omega_2(s,v)\,\mathrm{d}v. \tag{2.12}$$

This form of K reduces (2.7) to

$$\begin{aligned}\tilde{b}^{\pm}_{c_0,a,\beta}[g] = &\int_0^L\int_{-a}^{a}(1+u\gamma(s))^{-2}|g_s|^2\,\mathrm{d}u\,\mathrm{d}s + \int_0^L\int_{-a}^{a}|g_u|^2\,\mathrm{d}u\,\mathrm{d}s\\ &-\beta\int_0^L|g(s,0)|^2\,\mathrm{d}s - \frac{b_\pm}{2}\int_0^L\frac{\gamma(s)}{1+a\gamma(s)}|g(s,a)|^2\,\mathrm{d}s\\ &+\frac{b_\pm}{2}\int_0^L\frac{\gamma(s)}{1-a\gamma(s)}|g(s,-a)|^2\,\mathrm{d}s\\ &+\int_0^L\int_{-a}^{a}\big(\theta(s,u)c_0^2+(1+u\gamma(s))^{-2}K_s^2+K_u^2+V(s,u)\\ &+2c_0\Omega_1(s,u)K_s-2c_0\Omega_2(s,u)K_s\big)|g|^2\,\mathrm{d}u\,\mathrm{d}s\\ &+2\int_0^L\int_{-a}^{a}(c_0\Omega_1(s,u)+(1+u\gamma(s))^{-2}K_s)\,\mathrm{Im}\,\overline{g}g_s\,\mathrm{d}u\,\mathrm{d}s\end{aligned} \tag{2.13}$$

for $g\in\mathcal{Q}_a^{\pm}$, respectively.

Let us remark that because of the properties of the curve Γ, we have $\Omega_2(0,u) = \Omega_2(L,u)\ \forall u\in(-a,a)$. So the domains $\mathcal{Q}_a^{\pm}$ are not changed under the unitary operator M_{c_0}.

Let $\tilde{D}_{c_0,a,\beta}$ be the self-adjoint operators associated with the forms $\tilde{b}^{\pm}_{c_0,a,\beta}$, respectively. We have the following result:

Lemma 2.2. $M^*_{c_0}D^{\pm}_{c_0,a,\beta}M_{c_0} = \tilde{D}^{\pm}_{c_0,a,\beta}$.

In the estimation of the $\tilde{D}^{\pm}_{c_0,a,\beta}$, let us use the same notation as in [4]

$$\gamma_+ = \max_{[0,L]}|\gamma(\cdot)| \qquad N_{c_0}(a) = \max_{(s,u)\in[0,L]\times[-a,a]}2|c_0\Omega_1(s,u)+(1+u\gamma(s))^{-2}K_s|$$

and

$$M_{c_0}(a) := \max_{(s,u)\in[0,L]\times[-a,a]}\left|W_{c_0}(s,u)+\tfrac14\gamma(s)^2\right|$$

where

$$W_{c_0}(s,u) = \theta(s,u)c_0^2+(1+u\gamma(s))^{-2}K_s^2+K_u^2+V(s,u)+2c_0(\Omega_1(s,u)K_s-\Omega_2(s,u)K_s). \tag{2.14}$$

Since $c_0\in I$ and I is a compact interval, then there exists T such that $N_{c_0}(a)+M_{c_0}(a)\leqslant Ta$ for $0<a<\frac{1}{2\gamma_+}$ and $c_0\in I$, where T is independent of a and c_0. For fixed $0<a<\frac{1}{2\gamma_+}$, as in [4] we define

$$\begin{aligned}\hat{b}^{\pm}_{c_0,a,\beta}[g] = &\int_0^L\int_{-a}^{a}\Big(\big[(1\pm u\gamma_+)^{-2}\pm\tfrac12 N_{c_0}(a)\big]|\partial_s g|^2+|\partial_u g|^2\\ &+\big[-\tfrac14\gamma(s)^2\pm\tfrac12 N_{c_0}(a)\pm M_{c_0}(a)\big]|g|^2\Big)\,\mathrm{d}u\,\mathrm{d}s\\ &-\beta\int_0^L|g(s,o)|^2\,\mathrm{d}s-\gamma_+b_\pm\int_0^L(|g(s,a)|^2+|g(s,-a)|^2)\,\mathrm{d}s\end{aligned} \tag{2.15}$$

for $g\in\mathcal{Q}_a^{\pm}$, respectively. Since $|\mathrm{Im}(\overline{g}\partial_s g)|\leqslant\frac12(|g|^2+|\partial_s g|^2)$, we obtain

$$\tilde{b}^{+}_{c_0,a,\beta}[g]\leqslant\hat{b}^{+}_{c_0,a,\beta}[g] \qquad \text{for } g\in\mathcal{Q}_a^{+} \tag{2.16}$$

$$\hat{b}^{-}_{c_0,a,\beta}[g] \leqslant \tilde{b}^{-}_{c_0,a,\beta}[g] \qquad \text{for } g \in \mathcal{Q}^{-}_{a}. \tag{2.17}$$

Let $\hat{H}^{\pm}_{c_0,a,\beta}$ be the self-adjoint operators associated with the form $\hat{b}^{\pm}_{c_0,a,\beta}$, respectively. Furthermore, let $T^{+}_{a,\beta}$ be the self-adjoint operator associated with the form

$$t^{+}_{a,\beta}[f] = \int_{-a}^{a} |f'(u)|^2\,\mathrm{d}u - \beta|f(0)|^2 \qquad f \in H^1_0(]-a,a[)$$

and similarly, let $T^{-}_{a,\beta}$ be the self-adjoint operator associated with the form

$$t^{-}_{a,\beta}[f] = \int_{-a}^{a} |f'(u)|^2\,\mathrm{d}u - \beta|f(0)|^2 - \gamma_+\big(|f(a)|^2 + |f(-a)|^2\big) \qquad f \in H^1(]-a,a[).$$

As in [4], let us denote by $\mu^{\pm}_{j}(c_0,a)$ the jth eigenvalue of the following operator, define on $L^2(]0,L[)$, by

$$U^{\pm}_{a,\beta} = -\left[(1 \mp u\gamma_+)^{-2} \pm \frac{1}{2}N_{c_0}(a)\right]\frac{\mathrm{d}^2}{\mathrm{d}s^2} - \frac{1}{4}\gamma(s)^2 \pm \frac{1}{2}N_{c_0}(a) \pm M_{c_0}(a) \tag{2.18}$$

in $L^2((0,L))$ with the domain P_{c_0} specified in the previous section. Then we have

$$\hat{H}^{\pm}_{c_0,a,\beta} = U^{\pm}_{c_0,a} \otimes 1 + 1 \otimes T^{\pm}_{a,\beta}. \tag{2.19}$$

Let $\mu^{\pm}_{j}(c_0,a)$ be the jth eigenvalue of $U^{\pm}_{c_0,a}$ counted with multiplicity. We shall prove the following estimate as in [4].

Proposition 2.1. *Let $j \in \mathbb{N}$. Then there exists $C(j) > 0$ such that*

$$\left|\mu^{+}_{j}(c_0,a) - \mu_j(c_0)\right| + |\mu^{-}_{j}(c_0,a) - \mu_j(c_0)| \leqslant C(j)a$$

holds for $c_0 \in I$ and $0 < a < \frac{1}{2\gamma_+}$, where $C(j)$ is independent of c_0 and a.

Proof. Since

$$U^{+}_{c_0,a} - \left[(1-a\gamma_+)^{-2} + \frac{1}{2}N_{c_0}(a)\right]S_{c_0}$$
$$= \frac{1}{4}\left[\frac{a\gamma_+(2-a\gamma_+)}{(1-a\gamma_+)^2} + \frac{1}{2}N_{c_0}(a)\right]\gamma(s)^2 + \frac{1}{2}N_{c_0}(a) + M_{c_0}(a)$$

$N_{c_0}(a) + M_{c_0}(a) \leqslant Ta$ for $0 < a < \frac{1}{2\gamma_+}$ and $c_0 \in I$, we infer that there is a constant $C_1 > 0$ such that

$$\left\|U^{+}_{c_0,a} - \left[(1-a\gamma_+)^{-2} + \tfrac{1}{2}N_{c_0}(a)\right]S_{c_0}\right\| \leqslant C_1 a$$

for $0 < a < \frac{1}{2\gamma_+}$ and $c_0 \in I$. This together with the min–max principle implies that

$$\left|\mu^{+}_{j}(c_0,a) - \left[(1-a\gamma_+)^{-2} + \tfrac{1}{2}N_{c_0}(a)\right]\mu_j(c_0)\right| \leqslant C_1 a$$

for $0 < a < \frac{1}{2\gamma_+}$ and $c_0 \in I$. Since $\mu_j(\cdot)$ is continuous, we claim that there exists a constant $C_2 > 0$, such that

$$\left|\mu^{+}_{j}(c_0,a) - \mu_j(c_0)\right| \leqslant C_2 a$$

for $0 < a < \frac{1}{2\gamma_+}$ and $c_0 \in I$. In a similar way, we infer the existence of a constant $C_3 > 0$ such that

$$|\mu^{-}_{j}(c_0,a) - \mu_j(c_0)| \leqslant C_3 a$$

for $0 < a < \frac{1}{2\gamma_+}$ and $c_0 \in I$. □

Let us recall the following result from [3].

Proposition 2.2.

(a) Suppose that $\beta a > \frac{8}{3}$. Then $T^{+}_{a,\beta}$ has only one negative eigenvalue, which we denote by $\zeta_{a,\beta}$. It satisfies the inequality $-\frac{1}{4}\beta^2 < \zeta_{a,\beta} < -\frac{1}{4}\beta^2 + 2\beta^2 \exp\left(-\frac{1}{2}\beta\right)$.

(b) Let $\beta > 8$ and $\beta > 8/3\gamma_+$. Then $T^{-}_{a,\beta}$ has a unique negative eigenvalue $\zeta^{-}_{a,\beta}$, and moreover, we have $-\frac{1}{4}\beta^2 - \frac{2205}{16}\beta^2 \exp\left(-\frac{1}{2}\beta\right) < \zeta^{-}_{a,\beta} < -\frac{1}{4}\beta^2$.

Proof of theorem 2.1. We take $a(\beta) = 6\beta^{-1}\ln\beta$. Let $\xi^{\pm}_{\beta,j}$ be the jth eigenvalue of $T^{\pm}_{a(\beta),\beta}$, by proposition 2.2 we have

$$\xi^{\pm}_{\beta,1} = \zeta^{\pm}_{a(\beta),\beta} \qquad \xi^{\pm}_{\beta,2} \geqslant 0.$$

From decompositions (2.19) we infer that $\left\{\xi^{\pm}_{\beta,j} + \mu^{\pm}_{k}(B, a(\beta))\right\}_{j,k\in\mathbb{N}}$, properly ordered, is the sequence of the eigenvalues of $\hat{H}^{\pm}_{c_0,a(\beta),\beta}$ counted with multiplicity. Proposition 2.1 gives

$$\xi^{\pm}_{\beta,j} + \mu_k(c_0, a(\beta)) \geqslant \mu^{\pm}_{1}(c_0, a(\beta)) = \mu_1(c_0) + \mathcal{O}(\beta^{-1}\ln\beta) \tag{2.20}$$

for $c_0 \in I$, $j \geqslant 2$ and $k \geqslant 1$, where the error term is uniform with respect to $c_0 \in I$. For a fixed $j \in \mathbb{N}$, we take

$$\tau^{\pm}_{c_0,\beta,j} = \zeta^{\pm}_{a(\beta),\beta} + \mu^{\pm}_{j}(c_0, a(\beta)).$$

Combining propositions 2.1 and 2.2 we get

$$\tau^{\pm}_{c_0,\beta,j} = -\tfrac{1}{4}\beta^2 + \mu_j(c_0) + \mathcal{O}(\beta^{-1}\ln\beta) \qquad \text{as } \beta \to \infty \tag{2.21}$$

where the error term is uniform with respect to $c_0 \in I$. Let us fix $n \in \mathbb{N}$. Combining (2.20) with (2.21) we infer that there exists $\beta(n, I) > 0$ such that the inequalities

$$\tau^{+}_{c_0,\beta,n} < 0 \qquad \tau^{+}_{c_0,\beta,n} < \xi^{+}_{\beta,j} + \mu^{+}_{k}(c_0, a(\beta)) \qquad \tau^{-}_{c_0,\beta,n} < \xi^{-}_{\beta,j} + \mu^{-}_{k}(c_0, a(\beta))$$

hold for $c_0 \in I$, $\beta \geqslant \beta(n, I)$, $j \geqslant 2$ and $k \geqslant 1$. Hence the jth eigenvalue of $\hat{H}^{\pm}_{c_0,a(\beta),\beta}$ counted with multiplicity is $\tau^{\pm}_{c_0,\beta,j}$ for $c_0 \in I$, $j \leqslant n$, and $\beta \geqslant \beta(n, I)$. Let $c_0 \in I$ and $\beta \geqslant \beta(n, I)$. We denote by $\kappa^{\pm}_{j}(c_0, \beta)$ the jth eigenvalue of $L^{\pm}_{c_0,a,\beta}$. Combining our basic estimate and the result of [4] with lemmas 2.1 and 2.2, relations (2.16) and (2.17), and the min–max principle, we arrive at the inequalities

$$\tau^{-}_{c_0,\beta,j} \leqslant \kappa^{-}_{j}(c_0, \beta) \qquad \text{and} \qquad \kappa^{+}_{j}(c_0, \beta) \leqslant \tau^{+}_{c_0,\beta,j} \qquad \text{for } 1 \leqslant j \leqslant n \tag{2.22}$$

so we have $\kappa^{+}_{n}(c_0, \beta) < 0 < \inf\sigma_{\mathrm{ess}}(H_{c_0,\beta})$. Hence the min–max principle and the result of [5] imply that $H_{c_0,\beta}$ has at least n eigenvalues in $(-\infty, \kappa^{+}_{n}(c_0, \beta)]$. Given $1 \leqslant j \leqslant n$, we denote by $\lambda_j(c_0, \beta)$ the jth eigenvalue of $H_{c_0,\beta}$. It satisfies

$$\kappa^{-}_{j}(c_0, \beta) \leqslant \lambda_j(c_0, \beta) \leqslant \kappa^{+}_{j}(c_0, \beta) \qquad \text{for } 1 \leqslant j \leqslant n$$

this together with (2.21) and (2.22) implies that

$$\lambda_j(c_0, \beta) = -\tfrac{1}{4}\beta^2 + \mu_j(c_0) + \mathcal{O}(\beta^{-1}\ln\beta) \qquad \text{as } \beta \to \infty \qquad \text{for } 1 \leqslant j \leqslant n$$

where the error term is uniform with respect to $c_0 \in I$. This completes the proof. □

Proof of corollary 2.1. Theorem 2.1 with [5] (theorem XIII.89) yields the claim. □

3. Remarks

The essential of this paper is the determination of the unitary operator (2.6) which permits us to have all the conditions of [4], to have our results.

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